12 Equation of a line in three dimensions

Try this worksheet after you have completed section 12.4

In Chapter 12 you saw that the vector equation of a straight line has the form **r** = **a** + t**b** where **a** is the position vector of a point on the line and **b** is the direction vector of the line.

So the vector $\mathbf{r}=\begin{pmatrix}1\\3\\-2\end{pmatrix}+t\begin{pmatrix}2\\4\\-5\end{pmatrix}$ is a straight line through the point (1, 3, −2) in the direction of $\begin{pmatrix}2\\4\\-5\end{pmatrix}$

→ In general $\mathbf{r}=\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}+t\begin{pmatrix}l\\m\\n\end{pmatrix}$ is a straight line that passes through (x_0, y_0, z_0) in the direction of $\begin{pmatrix}l\\m\\n\end{pmatrix}$

This equation can be written in **parametric form** with the vector equation expressed in terms of the parameter t.

EXAMPLE 1

Write the vector equation $\mathbf{r}=\begin{pmatrix}1\\3\\-2\end{pmatrix}+t\begin{pmatrix}2\\4\\-5\end{pmatrix}$ in parametric form.

Answer

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\3\\-2\end{pmatrix}+t\begin{pmatrix}2\\4\\-5\end{pmatrix}=\begin{pmatrix}1+2t\\3+4t\\-2-5t\end{pmatrix}$$

Rewrite $\mathbf{r}=\begin{pmatrix}1\\3\\-2\end{pmatrix}+t\begin{pmatrix}2\\4\\-5\end{pmatrix}$

$x=1+2t$

$y=3+4t$

$z=-2-5t$

Equate the components to give the parametric equations.

From the parametric form of the equation you can obtain the **Cartesian equation** of the straight line.

EXAMPLE 2

Give the Cartesian equation of the line with vector equation $\mathbf{r}=\begin{pmatrix}1\\3\\-2\end{pmatrix}+t\begin{pmatrix}-1\\4\\-5\end{pmatrix}$

Answer

r can be written in parametric form as

Write **r** *in parametric form (as in Example 1).*

$x = 1 - t$

$y = 3 + 4t$

$z = -2 - 5t$

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$t = 1 - x$ $t = \frac{y-3}{4}$ $t = \frac{z+2}{-5}$	*Make t the subject in each of these equations.*
$1 - x = \frac{y-3}{4} = \frac{z+2}{-5}$	*Equate the expressions (they are all equal to t).*

This is the Cartesian equation of the line. It is the three-dimensional version of $y = mx + c$

→ The Cartesian equation of a line is $\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$

This line passes through (x_0, y_0, z_0) in the direction of $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$

EXAMPLE 3

Find the vector equation of the line with the Cartesian equation $\frac{x-2}{4} = \frac{y-1}{-2} = \frac{z+3}{2}$

Answer

The line passes through the point (2, 1, −3) with direction vector $4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

The vector equation of the line is therefore $\mathbf{r} = (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) + t(4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$

Exercise

1 Find the Cartesian equation of the line that is parallel to the vector $2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$ and passes though the point A with position vector $3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.

2 Find the Cartesian equation of the line that passes through the points (4, 0,−3) and (3, 2, 0).

3 Find the vector equation of the line with the Cartesian equation

a $\frac{x-2}{3} = \frac{y+4}{2} = \frac{z}{-3}$

b $\frac{x-1}{5} = \frac{y}{2} = \frac{2-z}{3}$

4 Find the acute angle between the two straight lines with Cartesian equations L_1 and L_2 given by

L_1: $\frac{x-3}{5} = y + 2 = \frac{z}{2}$

and

L_2: $\frac{x+1}{2} = \frac{y+1}{2} = \frac{z-3}{-5}$

Chapter 12 extension worked solutions

1 The Cartesian equation of the line is

$$\frac{x-3}{2}=\frac{y+2}{3}=\frac{z-2}{-5}$$

2 The direction vector of the line is $\begin{pmatrix}3\\2\\0\end{pmatrix}-\begin{pmatrix}4\\0\\-3\end{pmatrix}=\begin{pmatrix}-1\\2\\3\end{pmatrix}$

The Cartesian equation of the line is therefore $\frac{x-4}{-1}=\frac{y}{2}=\frac{z+3}{3}$ or $\frac{x-3}{-1}=\frac{y-2}{2}=\frac{z}{3}$

3 **a** If we write the line as $\frac{x-2}{3}=\frac{y-(-4)}{2}=\frac{z-0}{-3}$ then we can see that the line passes through the point (2, −4, 0) in the direction of $\begin{pmatrix}3\\2\\-3\end{pmatrix}$

b If we write the line as $\frac{x-1}{5}=\frac{y-0}{2}=\frac{z-2}{-3}$ then we can see that the line passes through the point (1, 0, 2) in the direction of $\begin{pmatrix}5\\2\\-3\end{pmatrix}$

4 If we write L_1: $\frac{x-3}{5}=y+2=\frac{z}{2}$ as $\frac{x-3}{5}=\frac{y-(-2)}{1}=\frac{z-0}{2}$ then the direction vector of L_1 is $\mathbf{a}=\begin{pmatrix}5\\1\\2\end{pmatrix}$

If we write L_2: $\frac{x+1}{2}=\frac{y+1}{2}=\frac{z-3}{-5}$ as $\frac{x-(-1)}{2}=\frac{y-(-1)}{2}=\frac{z-3}{-5}$ then the direction vector of L_2 is $\mathbf{b}=\begin{pmatrix}2\\2\\-5\end{pmatrix}$

The angle between the lines is the angle between the direction vectors

Using $\cos\theta=\frac{\mathbf{a}\,.\,\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$

$$\cos\theta=\frac{5\times2+1\times2+2\times-5}{\sqrt{5^2+1^2+2^2}\sqrt{2^2+2^2+(-5)^2}}$$

$$=\frac{2}{\sqrt{30}\sqrt{33}}$$

Therefore $\theta=\cos^{-1}\left(\frac{2}{\sqrt{30}\sqrt{33}}\right)$

And $\theta=86.4°$