14 More trigonometric derivatives and integrals

Try this worksheet after you have completed section 14.3

You should know these derivatives and integrals of trigonometric functions.

→ $f(x) = \sin x \Rightarrow f'(x) = \cos x$ $\qquad \int \sin x\,dx = -\cos x + C$

$f(x) = \cos x \Rightarrow f'(x) = -\sin x$ $\qquad \int \cos x\,dx = \sin x + C$

$f(x) = \tan x \Rightarrow f'(x) = \frac{1}{\cos^2 x}, \cos x \neq 0$

Exercise 1

1 Use that fact that $\tan x = \frac{\sin x}{\cos x}, \cos x \neq 0$ and $\int \frac{1}{u}\,du = \ln|u| + C$ to find $\int \tan x\,dx$.

Derivatives and integrals of reciprocal functions

Each of the three trigonometric functions has a reciprocal trigonometric function.

→ The reciprocal of sine is **cosecant**. $\qquad \csc x = \frac{1}{\sin x}, \sin x \neq 0$

The reciprocal of cosine is **secant**. $\qquad \sec x = \frac{1}{\cos x}, \cos x \neq 0$

The reciprocal of tangent is **cotangent**.
$\cot x = \frac{1}{\tan x}, \tan x \neq 0$ or $\cot x = \frac{\cos x}{\sin x}, \sin x \neq 0$

Exercise 2

1 Express the derivative of $\tan x$ in terms of a reciprocal trigonometric function.

2 Use that fact that $\csc x = \frac{1}{\sin x}$ to show that the derivative of $\csc x$ is $-\csc x \cot x$.

3 Use that fact that $\sec x = \frac{1}{\cos x}$ to show that the derivative of $\sec x$ is $\sec x \tan x$.

4 Use that fact that $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$ to find the derivative of $\cot x$.

5 Use that fact that $\cot x = \frac{\cos x}{\sin x}$ and $\int \frac{1}{u}\,du = \ln|u| + C$ to find $\int \cot x\,dx$.

Integrals of secant and cosecant

Deriving the integrals of cosecant and secant takes a little creativity.

Exercise 3

1 Show that $\int \csc x \, dx = -\ln|\csc x + \cot x| + C$. To do this, follow these hints.

a Rewrite $\int \csc x \, dx$ as $\int (\csc x)(1) \, dx$

b Substitute $\frac{\csc x + \cot x}{\csc x + \cot x}$ in place of the factor of 1.

c Distribute.

d Integrate using a u-substitution where $u = \csc x + \cot x$.

2 Use a strategy similar to that in Question 1 to find $\int \sec x \, dx$.

3 Make a summary of the derivatives and integrals of the three basic trigonometric functions sine, cosine and tangent and their three reciprocal functions cosecant, secant and cotangent.

Use the fact that you can now differentiate and integrate all six trigonometric functions to solve the problems in Exercise 4.

Exercise 4

1 Find the derivative of each function.

a $f(x) = 2\tan 3x$

b $y = \sec\left(x^2\right)$

c $f(x) = \cot(\pi x)$

d $g(x) = \csc x \cot x$

e $f(x) = \frac{\sec x}{1 + \tan x}$

2 Find each indefinite integral.

a $\int \csc 4x \, dx$

b $\int e^x \tan(e^x) \, dx$

c $\int \frac{\sec^2 x}{\tan x} \, dx$

d $\int \cot \frac{x}{2} \, dx$

e $\int \frac{\sin^2 x + \cos^2 x}{\cos x} \, dx$

Chapter 14 extension worked solutions

Exercise 1

1 $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$ *Use to identity $\tan x = \frac{\sin x}{\cos x}$ to rewrite the integrand.*

$= \int \frac{-\frac{du}{dx}}{u} dx$ *Use the substitution method to integrate. Let $u = \cos x$, then $\frac{du}{dx} = -\sin x$ and so $-\frac{du}{dx} = \sin x$.*

$= -\int \frac{1}{u} du$

$= -\ln|u| + C$

$= -\ln|\cos x| + C$

Exercise 2

1 $\frac{d}{dx}[\tan x] = \frac{1}{\cos^2 x}$

$= \left(\frac{1}{\cos x}\right)^2$ *Use the identity $\sec x = \frac{1}{\cos x}$*

$= (\sec x)^2$

$\frac{d}{dx}[\tan x] = \sec^2 x$

2 $\frac{d}{dx}[\csc x] = \frac{d}{dx}\left[\frac{1}{\sin x}\right]$ *Use the identity $\csc x = \frac{1}{\sin x}$*

$= \frac{(\sin x)(0) - (1)(\cos x)}{\sin^2 x}$ *Apply the quotient rule.*

$= -\frac{\cos x}{\sin^2 x}$

$= -\left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)$ *Factorize and then use the identities*

$\frac{d}{dx}[\csc x] = -\csc x \cot x$ *$\csc x = \frac{1}{\sin x}$ and $\cot x = \frac{\cos x}{\sin x}$*

3 $\frac{d}{dx}[\sec x] = \frac{d}{dx}\left[\frac{1}{\cos x}\right]$ *Use the identity $\sec x = \frac{1}{\cos x}$*

$= \frac{(\cos x)(0) - (1)(-\sin x)}{\cos^2 x}$ *Apply the quotient rule.*

$= \frac{\sin x}{\cos^2 x}$

$= \left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right)$ *Factorize and then use the identities*

$= \sec x \tan x$ *$\sec x = \frac{1}{\cos x}$ and $\tan x = \frac{\sin x}{\cos x}$*

4 $\frac{d}{dx}[\cot x] = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right]$ *Use the identity $\cot x = \frac{\cos x}{\sin x}$*

$= \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x}$ *Apply the quotient rule.*

$= \frac{-\left(\sin^2 x + \cos^2 x\right)}{\sin^2 x}$

$\frac{d}{dx}[\cot x] = -\frac{1}{\sin^2 x}$ or $-\csc^2 x$ *Use the identities $\sin^2 x + \cos^2 x = 1$ and $\csc x = \frac{1}{\sin x}$*

5 $\int \cot x\,dx = \int \frac{\cos x}{\sin x}dx$ *Use the identity $\cot x = \frac{\cos x}{\sin x}$ to rewrite the integrand*

$= \int \frac{\frac{du}{dx}}{u}dx$ *Use the substitution method to integrate. Let $u = \sin x$, then $\frac{du}{dx} = \cos x$*

$= \int \frac{1}{u}du$

$= \ln|u| + C$

$= \ln|\sin x| + C$

Exercise 3

1 a $\int \csc x\,dx = \int (\csc x)(1)\,dx$

b $\int (\csc x)(1)\,dx = \int (\csc x)\left(\frac{\csc x + \cot x}{\csc x + \cot x}\right)dx$

c $\int (\csc x)\left(\frac{\csc x + \cot x}{\csc x + \cot x}\right)dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}dx$

d $\int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}dx = -\int \frac{\frac{du}{dx}}{u}dx$ *Let $u = \csc x + \cot x$, then $\frac{du}{dx} = -\csc x \cot x - \csc^2 x$ or $-\frac{du}{dx} = \csc^2 x + \csc x \cot x$*

$= -\int \frac{1}{u}du$

$= -\ln|u| + C$

$= -\ln|\csc x + \cot x| + C$

2 $\int \sec x\,dx = \int (\sec x)\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)dx$ *Multiply by 1 where $1 = \frac{\sec x + \tan x}{\sec x + \tan x}$*

$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx$

$= \int \frac{\frac{du}{dx}}{u}dx$ *Let $u = \sec x + \tan x$, then $\frac{du}{dx} = \sec x \tan x + \sec^2 x = \sec^2 x + \sec x \tan x$*

$= \int \frac{1}{u}du$

$= \ln|u| + C$

$= \ln|\sec x + \tan x| + C$

3

$\frac{d}{dx}[\sin x] = \cos x$ $\quad$ $\frac{d}{dx}[\csc x] = -\csc x \cot x$

$\frac{d}{dx}[\cos x] = -\sin x$ $\quad$ $\frac{d}{dx}[\sec x] = \sec x \tan x$

$\frac{d}{dx}[\tan x] = \frac{1}{\cos^2 x}$ or $\sec^2 x$ $\quad$ $\frac{d}{dx}[\cot x] = -\frac{1}{\sin^2 x}$ or $-\csc^2 x$

$\int \sin x\,dx = -\cos x + C$ $\quad$ $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

$\int \cos x\,dx = \sin x + C$ $\quad$ $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

$\int \tan x\,dx = -\ln|\cos x| + C$ $\quad$ $\int \cot x\,dx = \ln|\sin x| + C$

Exercise 4

1 **a** $f(x) = 2\tan 3x$

$f'(x) = 2(\sec^2 3x)(3) = 6\sec^2 3x$

b $y = \sec(x^2)$

$\frac{dy}{dx} = [\sec(x^2)\tan(x^2)](2x) = 2x\sec(x^2)\tan(x^2)$

c $f(x) = \cot(\pi x)$

$f'(x) = [-\csc^2(\pi x)](\pi) = -\pi\csc^2(\pi x)$

d $g(x) = \csc x \cot x$

$g'(x) = (\csc x)(-\csc^2 x) + (\cot x)(-\csc x \cot x) = -\csc^3 x - \csc x \cot^2 x$

e $f(x) = \frac{\sec x}{1+\tan x}$

$f'(x) = \frac{(1+\tan x)(\sec x \tan x) - (\sec x)(\sec^2 x)}{(1+\tan x)^2} = \frac{\sec x \tan x + \sec x \tan^2 x - \sec^3 x}{(1+\tan x)^2}$

2 **a** $\int \csc 4x \, dx = -\frac{1}{4}\ln|\csc 4x + \cot 4x| + C$

b Let $u = e^x$ then $\frac{du}{dx} = e^x$

$\int e^x \tan(e^x)\,dx = \int \frac{du}{dx}\tan(u)\,dx = \int \tan u \, du = -\ln|\cos u| + C = -\ln|\cos e^x| + C$

c Let $u = \tan x$ then $\frac{du}{dx} = \sec^2 x$

$\int \frac{\sec^2 x}{\tan x}\,dx = \int \frac{\frac{du}{dx}}{u}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\tan x| + C$

d $\int \cot\frac{x}{2}\,dx = \int \cot\left(\frac{1}{2}x\right)dx = \frac{1}{\frac{1}{2}}\ln\left|\sin\left(\frac{1}{2}x\right)\right| + C = 2\ln\left|\sin\frac{x}{2}\right| + C$

e $\int \frac{\sin^2 x + \cos^2 x}{\cos x}\,dx = \int \frac{1}{\cos x}\,dx = \int \sec x \, dx = \ln|\sec x + \tan x| + C$