3 Conditional probability

Try this worksheet after you have completed Exercise 3I.

The conditional probability law in Chapter 3 related events A and B such that

→ $P(A|B) = \frac{P(A \cap B)}{P(B)}$ (1) and

$P(B|A) = \frac{P(B \cap A)}{P(A)}$ (2)

Bayes' theorem

You can use formulae (1) and (2) to find the probability of an event happening given that another has already happened.

Rearrange (1) to get $P(A \cap B) = P(A|B)P(B)$ (3)

and (2) to get $P(B \cap A) = P(B|A)P(A)$ (4)

Since $(A \cap B) = (B \cap A)$, then $P(A \cap B) = P(B \cap A)$
Therefore from (3) and (4)

$P(A|B)P(B) = P(B|A)P(A)$

Rearrange this to give $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$ (5)

Now suppose that instead of a single event A there were n alternative previous events that could have happened, namely $A_1, A_2, A_3, \ldots, A_n$.

These events are mutually exclusive and exhaustive so that $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = S$, the complete sample space, and B is an arbitrary event of S.

Then for $i = 1, 2, 3, \ldots, n$

→ $P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \ldots + P(B|A_n)P(A_n)}$

This formula is known as Bayes' theorem.

You can use Bayes' theorem to 'reverse the conditions' in a problem.

EXAMPLE 1

A football team plays 60% of its games at home and 40% away. The team wins 8 out of 10 of its home games and 5 out of 10 of its away games.

a Find the probability that the team wins on a given Saturday.

b If the team wins on a certain Saturday, what is the probability that it played at home?

Answers

Let A_1 be the event that the game is played at home.

Let A_2 be the event that the game is played away.

Let B be the event that the team wins its game.

▶ Continued on next page

Events A_1 and A_2 are mutually exclusive and exhaustive.

$P(A_1) = 0.6$, $P(A_2) = 0.4$, $P(B|A_1) = 0.8$, $P(B|A_2) = 0.5$

a P(team wins the game) = P(B)

$$P(B) = P(B | A_1)P(A_1) + P(B | A_2)P(A_2)$$
$$P(B) = 0.8 \times 0.6 + 0.5 \times 0.4$$
$$= 0.68$$

So the probability that the team wins is 0.68

b From Bayes' theorem

$$P(A_1 | B) = \frac{P(B | A_1)P(A_1)}{P(B | A_1)P(A_1) + P(B | A_2)P(A_2)}$$

So

$$P(A | B) = \frac{0.8 \times 0.6}{0.68}$$
$$= \frac{12}{17} = 0.701$$

The denominator $P(B | A_1)P(A_1) + P(B | A_2)P(A_2)$ is the probability that the team wins wherever it plays = 0.68

Exercise

1 In a school 56% of the students are female. 10% of the female students and 5% of male students are left-handed.

- **a** What is the probability that a randomly chosen student is left-handed?
- **b** Given that a randomly chosen student is left-handed find the probability that the student is male.
- **c** Given that a randomly chosen student is not left-handed find the probability that the student is female.

2 A product tester rated 75% of products as satisfactory and 25% as unsatisfactory. Of the satisfactory products, 80% came from a particular company. Of the unsatisfactory products, 40% came from that company. If a product from the company is chosen at random, what is the probability that it will be rated as satisfactory?

3 Suppose there is a certain disease randomly found in one-half of one percent (0.005%) of the general population. A blood test for the disease is 99% effective in detecting the presence of this disease: that is, it will yield an accurate positive result in 99% of the cases where the disease is actually present. But it also yields 'false-positive' results in 5% of the cases where the disease is not present. Let A be the event that the disease is present in any particular person. Let B be the event that the test will yield a positive test result. What is the probability that a positive test result will be a true positive?

4 Three people work in a fruit-packing factory. Alice packs 40% of the fruit, Bruce packs 35% and Cathy packs 25%. The probability that Alice packs bad fruit is 0.05. The respective probabilities for Bruce and Cathy are 0.1 and 0.15. What is the probability that a box with bad fruit in it was packed by Alice?

5 In a test the probabilities that three pupils, Xavier, Yann and Zachariah solve the last problem on the paper are $\frac{2}{5}$, $\frac{3}{4}$ and $\frac{1}{3}$, respectively.

a Calculate the probability that the teacher marks from these candidates

i one, and only one, correct answer

ii not more than one correct answer

iii least one correct answer.

b Given that the teacher receives exactly one correct solution, determine the probability that this answer was provided by Xavier.

Chapter 3 extension worked solutions

1 Let A_1 be the event that the student is female.
Let A_2 be the event that the student is male.
Let L be the event that the student is left-handed.
Events A_1 and A_2 are mutually exclusive and exhaustive.
$P(A_1) = 0.56$, $P(A_2) = 0.44$, $P(L|A_1) = 0.1$, $P(L|A_2) = 0.05$

a P(left-handed) = P(L)
now using
$$P(L) = P(L \mid A_1)P(A_1) + P(L \mid A_2)P(A_2)$$
$$P(L) = 0.1 \times 0.56 + 0.05 \times 0.44$$
$$= 0.078$$
So the probability that a student is left-handed is 0.078

b From Bayes' theorem
$$P(A_2 \mid L) = \frac{P(L \mid A_2)P(A_2)}{P(L \mid A_2)P(A_2) + P(L \mid A_1)P(A_1)}$$

The denominator = P(L)

So
$$P(A_2 \mid L) = \frac{0.05 \times 0.44}{0.078}$$
$$= 0.282$$

c $$P(A_1 \mid L') = \frac{P(L' \mid A_1)P(A_1)}{P(L' \mid A_1)P(A_1) + P(L' \mid A_2)P(A_2)}$$
So
$$P(A_1 \mid L') = \frac{0.9 \times 0.56}{0.9 \times 0.56 + 0.95 \times 0.44}$$
$$= 0.547$$

2 Let A be the event that the product came from the company.
Let S be the event that the product is satisfactory.
Let S' be the event that the product is not satisfactory.
Events S and S' are mutually exclusive and exhaustive.
$P(S) = 0.75$, $P(S') = 0.25$, $P(A \mid S) = 0.8$, $P(A \mid S') = 0.4$

a $$P(S \mid A) = \frac{P(A \mid S)P(S)}{P(A \mid S)P(S) + P(A \mid S')P(S')}$$
So
$$P(S \mid A) = \frac{0.8 \times 0.75}{0.8 \times 0.75 + 0.4 \times 0.25}$$
$$= 0.857$$

3 Let A represent the event that the patient has the disease, $P(A) = 0.005$
Let B represent the evidence of a positive test result, $P(B) = 0.99$
The probability that the patient actually has the disease given the positive test result is given by $P(A \mid B)$.
$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A')P(A')}$$
$$= \frac{0.99 \times 0.005}{0.99 \times 0.005 + 0.05 \times 0.995}$$
$$= 0.0905$$

4 Let A be the event that the box is packed by Alice.
Let B be the event that the box is packed by Bruce.
Let C be the event that the box is packed by Cathy.
Let D be the event that a selected box contains bad fruit.
$P(A) = 0.4$, $P(B) = 0.35$, $P(C) = 0.\ 25$, $P(D \mid A) = 0.05$, $P(D \mid B) = 0.1$, $P(D \mid C) = 0.15$
You need to find $P(A|D)$.
$$P(A \mid D) = \frac{P(D \mid A)P(A)}{P(D \mid A)P(A) + P(D \mid B)P(B) + P(D \mid C)P(C)}$$
So
$$P(A \mid D) = \frac{0.05 \times 0.4}{0.05 \times 0.4 + 0.1 \times 0.35 + 0.15 \times 0.25}$$
$$= 0.216$$

5 **a** Let X be the event that the correct answer was given by Xavier.
Let Y be the event that the correct answer was given by Yann.

Let Z be the event that the correct answer was given by Zachariah.

i The probability that there is one, and only one, correct answer
$= P(X \cap Y' \cap Z') + P(X' \cap Y \cap Z') + P(X' \cap Y' \cap Z)$

$$= \left(\frac{2}{5}\times\frac{1}{4}\times\frac{2}{3}\right)+\left(\frac{3}{5}\times\frac{3}{4}\times\frac{2}{3}\right)+\left(\frac{3}{5}\times\frac{3}{4}\times\frac{1}{3}\right)=\frac{4+18+9}{60}=\frac{31}{60}$$

ii Probability that there is not more than one correct answer
= P(no correct answers) + P(exactly one correct answer)
P(no correct answers) $= P(X' \cap Y' \cap Z')$

$$= \left(\frac{3}{5}\times\frac{1}{4}\times\frac{2}{3}\right)=\frac{6}{60}=\frac{1}{10}$$

P(exactly one correct answer) $= \frac{31}{60}$

so P(not more than one correct answer) $= \frac{31}{60}+\frac{6}{60}=\frac{37}{60}$

iii P(at least one correct answer)
= 1 – P(no correct answers)

$$=1-\frac{1}{10}=\frac{9}{10}$$

b Let C be the event exactly one answer is correct.
The probability that Xavier provided this answer $= P(X \mid C)$

$$P(X \mid C)=\frac{P(C \mid X)P(X)}{P(C \mid X)P(X)+P(C \mid Y)P(Y)+P(C \mid Z)P(Z)}$$

So

$$P(X \mid C)=\frac{\left(\frac{2}{5}\times\frac{1}{4}\times\frac{2}{3}\right)}{\left(\frac{2}{5}\times\frac{1}{4}\times\frac{2}{3}\right)+\left(\frac{3}{5}\times\frac{3}{4}\times\frac{2}{3}\right)+\left(\frac{3}{5}\times\frac{3}{4}\times\frac{1}{3}\right)}$$

$$=\frac{4}{60}\div\frac{31}{60}=\frac{4}{31}$$