7 An algebraic look at limits

Try this worksheet after you have completed section 7.1

Section 7.1 examined the concept of the limit of a function graphically and by using tables of values. This worksheet looks at finding some limits algebraically, using limit properties.

We know that $\lim_{x \to c} f(x)$ does not depend on the value of f at $x = c$. However, for some well-behaved functions $\lim_{x \to c} f(x) = f(c)$. In this case the limit may be found by **direct substitution**, that is by evaluating f at c.

Basic limit properties

Let b and c be real numbers and let n be a positive integer.

1 $\lim_{x \to c} b = b$ The limit of the constant function $f(x) = b$ as x approaches c is b.

2 $\lim_{x \to c} x = c$ The limit of the identify function $f(x) = x$ as x approaches c is c.

3 $\lim_{x \to c} x^n = c^n$ The limit of $f(x) = x^n$ where n is a positive integer, as x approaches c can be found by using direct substitution.

EXAMPLE 1

Find each limit.

a $\lim_{x \to 4} 2$ **b** $\lim_{x \to -3} x$ **c** $\lim_{x \to 3} x^2$

Answers

a $\lim_{x \to 4} 2 = 2$	*by Property 1*
b $\lim_{x \to -3} x = -3$	*by Property 2*
c $\lim_{x \to 3} x^2 = 3^2 = 9$	*by Property 3*

More limit properties

Let b and c be real numbers and let n be a positive integer.

Let $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$.

4 $\lim_{x \to c} [bf(x)] = bL$ The limit of a constant times a function is the constant times the limit of the function.

5 $\lim_{x \to c} [f(x) \pm g(x)] = L \pm M$ The limit of a sum (or difference) is the sum (or difference) of the limits.

6 $\lim_{x \to c} [f(x)g(x)] = LM$ The limit of a product is the product of the limits.

7 $\lim_{x \to c} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}$, if $M \neq 0$ The limit of a quotient is the quotient of the limits, if the denominator does not equal 0.

EXAMPLE 2

Find each limit.

a $\lim\limits_{x\to 2}(3x^2-2)$ **b** $\lim\limits_{x\to -1}\frac{2x}{x+3}$

Answers

a $\lim\limits_{x\to 2}(3x^2-2)=\lim\limits_{x\to 2}3x^2-\lim\limits_{x\to 2}2$	*by Property 5*
$=3\lim\limits_{x\to 2}x^2-\lim\limits_{x\to 2}2$	*by Property 4*
$=3(2^2)-2$	*by Properties 3 and 1*
$=10$	*simplify*
b $\lim\limits_{x\to -1}\frac{2x}{x+3}=\frac{\lim\limits_{x\to -1}2x}{\lim\limits_{x\to -1}(x+3)}$	*by Property 7*
$=\frac{2\lim\limits_{x\to -1}x}{\lim\limits_{x\to -1}x+\lim\limits_{x\to -1}3}$	*by Property 4* *by Property 5*
$=\frac{2(-1)}{-1+3}$	*by Property 2* *by Properties 2 and 1*
$=-1$	*simplify*

These examples show that you can use direct substitution to find limits of polynomials and limits of rational functions that do not lead to zero.

For example:

$$\lim_{x\to 2}(3x^2-2)=3(2^2)-2=10 \text{ and } \lim_{x\to -1}\frac{2x}{x+3}=\frac{2(-1)}{-1+3}=-1.$$

Sometimes when direct substitution leads to a zero in the denominator of a rational function, you can factorize and then rewrite the function. You can then use direct substitution.

EXAMPLE 3

Find $\lim\limits_{x\to 2}\frac{x-2}{x^2-4}$

Answer

$\lim\limits_{x\to 2}\frac{x-2}{x^2-4}$	*Direct substitution leads to 0 in the denominator*
$\lim\limits_{x\to 2}\frac{x-2}{x^2-4}=\lim\limits_{x\to 2}\frac{x-2}{(x-2)(x+2)}=\lim\limits_{x\to 2}\frac{1}{x+2}=\frac{1}{2+2}=\frac{1}{4}$	*Factorize the denominator, simplify and then substitute*

Exercise

Find each limit. Show how the properties verify that direct substitution leads to the limit.

Simplify first where necessary.

1 $\lim\limits_{x\to -4}x$ **2** $\lim\limits_{x\to 1}\frac{x+2}{x-5}$ **3** $\lim\limits_{x\to 2}(2x^2+x-1)$

4 $\lim\limits_{x\to 2}\frac{x^2-4x+4}{x-2}$ **5** $\lim\limits_{x\to 6}5$ **6** $\lim\limits_{x\to -2}3x$

7 $\lim\limits_{x\to 3}\frac{x-3}{x+3}$ **8** $\lim\limits_{x\to -2}\frac{x^2-x-6}{x^2+6x+8}$ **9** $\lim\limits_{x\to 3}[(x^2-3)(x+4)]$

Chapter 7 extension worked solutions

Exercise

1 $\lim\limits_{x\to-4} x = -4$ (by Property 2)

2
$$\lim_{x\to1}\frac{x+2}{x-5} = \frac{\lim\limits_{x\to1}(x+2)}{\lim\limits_{x\to1}(x-5)} \quad \text{(by Property 7)}$$
$$= \frac{\lim\limits_{x\to1} x + \lim\limits_{x\to1} 2}{\lim\limits_{x\to1} x - \lim\limits_{x\to1} 5} \quad \text{(by Property 5)}$$
$$= \frac{1+2}{1-5} \quad \text{(by Properties 2 and 1)}$$
$$= -\frac{3}{4} \quad \text{(simplify)}$$

3
$$\lim_{x\to2}(2x^2+x-1) = \lim_{x\to2} 2x^2 + \lim_{x\to2} x - \lim_{x\to2} 1 \quad \text{(by Property 5)}$$
$$= 2\lim_{x\to2} x^2 + \lim_{x\to2} x - \lim_{x\to2} 1 \quad \text{(by Property 4)}$$
$$= 2(2^2)+2-1 \quad \text{(by Properties 3, 2 and 1)}$$
$$= 9 \quad \text{(simplify)}$$

4
$$\lim_{x\to2}\frac{x^2-4x+4}{x-2} = \lim_{x\to2}\frac{(x-2)^2}{x-2} \quad \text{(direct substitution leads to 0 in denominator; factor)}$$
$$= \lim_{x\to2}(x-2) \quad \text{(simplify)}$$
$$= \lim_{x\to2} x - \lim_{x\to2} 2 \quad \text{(by Property 5)}$$
$$= 2-2 \quad \text{(by Properties 2 and 1)}$$
$$= 0 \quad \text{(simplify)}$$

5 $\lim\limits_{x\to6} 5 = 5$ (by Property 1)

6
$$\lim_{x\to-2} 3x = 3\lim_{x\to-2} x \quad \text{(by Property 4)}$$
$$= 3(-2) \quad \text{(by Property 2)}$$
$$= -6 \quad \text{(simplify)}$$

7
$$\lim_{x\to3}\frac{x-3}{x+3} = \frac{\lim\limits_{x\to3}(x-3)}{\lim\limits_{x\to3}(x+3)} \quad \text{(by Property 7)}$$
$$= \frac{\lim\limits_{x\to3} x - \lim\limits_{x\to3} 3}{\lim\limits_{x\to3} x + \lim\limits_{x\to3} 3} \quad \text{(by Property 5)}$$
$$= \frac{3-3}{3+3} \quad \text{(by Properties 2 and 1)}$$
$$= 0 \quad \text{(simplify)}$$

8
$$\lim_{x\to-2}\frac{x^2-x-6}{x^2+6x+8} = \lim_{x\to-2}\frac{(x-3)(x+2)}{(x+4)(x+2)} \quad \text{(direct substitution leads to 0 in denominator; factor)}$$
$$= \lim_{x\to-2}\frac{x-3}{x+4} \quad \text{(simplify)}$$
$$= \frac{\lim\limits_{x\to-2}(x-3)}{\lim\limits_{x\to-2}(x+4)} \quad \text{(by Property 7)}$$
$$= \frac{\lim\limits_{x\to-2} x - \lim\limits_{x\to-2} 3}{\lim\limits_{x\to-2} x + \lim\limits_{x\to-2} 4} \quad \text{(by Property 5)}$$
$$= \frac{-2-3}{-2+4} \quad \text{(by Properties 2 and 1)}$$
$$= -\frac{5}{2} \quad \text{(simplify)}$$

9
$$\lim_{x\to3}\left[(x^2-3)(x+4)\right] = \lim_{x\to3}(x^2-3)\lim_{x\to3}(x+4) \quad \text{(by Property 6)}$$
$$= \left[\lim_{x\to3} x^2 - \lim_{x\to3} 3\right]\left[\lim_{x\to3} x + \lim_{x\to3} 4\right] \quad \text{(by Property 5)}$$
$$= \left[3^2-3\right]\left[3+4\right] \quad \text{(by Properties 3, 1, 2 and 1)}$$
$$= 42 \quad \text{(simplify)}$$