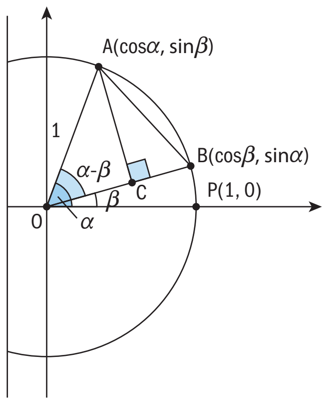
11 Angle sums and differences

Try this worksheet after you have completed section 11.3

You can use what you have learned about the unit circle and about triangle trigonometry to help you understand some important trigonometric formulae.
The angles α, β and $\alpha-\beta$, are shown on the unit circle in the diagram.



In triangle AOB, $AC \perp OB$.

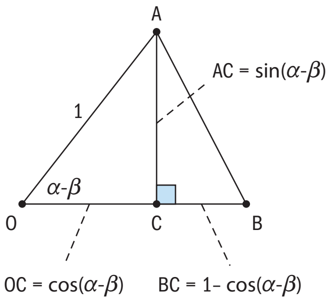
You can find AB using the distance formula:

$$AB=\sqrt{(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2}$$

$AB^2 = \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta$

$AB^2 = 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$ (using Pythagorean Identity from section 11.3)

$AB^2 = 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$



This is the same triangle AOB, with $AC \perp OB$

You can find AB using Pythagoras.

$$AB=\sqrt{AC^2+BC^2}$$

$AC^2 = 1 - OC^2 = 1 - \cos(\alpha-\beta)^2$ and $BC^2 = AB^2 - AC^2$

$$AB=\sqrt{(\sin(\alpha-\beta))^2+(1-\cos(\alpha-\beta))^2}$$

$AB^2 = \sin^2(\alpha-\beta) + 1 - 2\cos(\alpha-\beta) + \cos^2(\alpha-\beta)$

$AB^2 = 2 - 2\cos(\alpha-\beta)$

$\sin^2\theta+\cos^2\theta=1$

You now have two different expressions for AB^2, so you can set these equal to each other.

$2 - 2\cos(\alpha-\beta) = 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$\rightarrow \cos(\alpha-\beta) = (\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

You can use this formula to find the exact values of trigonometric ratios for angles such as 15°.

EXAMPLE 1

Find the exact value of cos 15°

Answer

$$\cos 15° = \cos (45-30)°$$
$$= \cos 45° \cos 30° + \sin 45° \sin 30°$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$\cos 15° = \frac{\sqrt{6}+\sqrt{2}}{4}$$

15° = 45° − 30°

There are three more formulae for finding the sines and cosines of other angle sums and differences.

Here are all four formulae.

→ $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ $\quad$ $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ $\quad$ $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Exercise 1

For questions 1 – 6 use the angle sum and difference formulae to find:

1 sin 15° $\quad$ **2** cos 75°

3 sin 105° $\quad$ **4** sin 75°

5 cos 105° $\quad$ **6** sin 165°

EXAMPLE 2

A and *B* are two acute angles.

$\sin A = \frac{2}{3}$ and $\cos B = \frac{1}{4}$

Find the exact value of sin (*A* + *B*).

Answer

$$\sin^2 A + \cos^2 A = 1$$
$$\left(\frac{2}{3}\right)^2 + \cos^2 A = 1$$
$$\cos A = \sqrt{1-\left(\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}$$

Use the formula $\sin^2\theta + \cos^2\theta = 1$ to find the values of cos A and sin B.

$$\sin^2 B + \cos^2 B = 1$$
$$\sin^2 B + \left(\frac{1}{4}\right)^2 = 1$$
$$\sin B = \sqrt{1-\left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$$

Use the formula again to find sin(A + B).

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$= \left(\frac{2}{3}\right)\left(\frac{1}{4}\right)+\left(\frac{\sqrt{5}}{3}\right)\left(\frac{\sqrt{15}}{4}\right)$$
$$= \frac{2+5\sqrt{3}}{12}$$

Exercise 2

For questions 1–6 let A and B be two acute angles such that $\cos A = \frac{2}{5}$ and $\sin B = \frac{1}{3}$.

Find the exact value of:

1 $\sin(A + B)$ **2** $\cos(A + B)$

3 $\sin(A - B)$ **4** $\cos(B - A)$

5 $\sin(2A)$ **6** $\cos(2B)$

Remember that sin(2A) can also be written as sin(A + A).

Chapter 11 extension worked solutions

Exercise 1

1 $\sin 15° = \sin(45-30)°$
$$= \sin 45° \cos 30° - \cos 45° \sin 30°$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$$

2 $\cos 75° = \cos(45+30)°$
$$= \cos 45° \cos 30° - \sin 45° \sin 30°$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$$

3 $\sin 105° = \sin(45+60)°$
$$= \sin 45° \cos 60° + \cos 45° \sin 60°$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{2}+\sqrt{6}}{4}$$

4 $\sin 75° = \sin(45+30)°$
$$= \sin 45° \cos 30° + \cos 45° \sin 30°$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$$

5 $\cos 105° = \cos(45+60)°$
$$= \cos 45° \cos 60° - \sin 45° \sin 60°$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{2}-\sqrt{6}}{4}$$

6 $\sin 165° = \sin(45+120)°$
$$= \sin 45° \cos 120° + \cos 45° \sin 120°$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Exercise 2

1 $\sin A = \sqrt{1-\left(\frac{2}{5}\right)^2} = \sqrt{\frac{21}{25}} = \frac{\sqrt{21}}{5}$

$\cos B = \sqrt{1-\left(\frac{1}{3}\right)^2} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$
$$= \left(\frac{\sqrt{21}}{5}\right)\left(\frac{2\sqrt{2}}{3}\right) + \left(\frac{2}{5}\right)\left(\frac{1}{3}\right) = \frac{2+2\sqrt{42}}{15}$$

2 $\cos(A+B) = \cos A \cos B - \sin A \sin B$
$$= \left(\frac{2}{5}\right)\left(\frac{2\sqrt{2}}{3}\right) - \left(\frac{\sqrt{21}}{5}\right)\left(\frac{1}{3}\right) = \frac{4\sqrt{2}-\sqrt{21}}{15}$$

3 $\sin(A-B) = \sin A \cos B - \cos A \sin B$
$$= \left(\frac{\sqrt{21}}{5}\right)\left(\frac{2\sqrt{2}}{3}\right) - \left(\frac{2}{5}\right)\left(\frac{1}{3}\right) = \frac{2\sqrt{42}-2}{15}$$

4 $\cos(B-A)=\cos B\cos A+\sin B\sin A$

$$=\left(\frac{2\sqrt{2}}{3}\right)\left(\frac{2}{5}\right)+\left(\frac{1}{3}\right)\left(\frac{\sqrt{21}}{5}\right)=\frac{4\sqrt{2}+\sqrt{21}}{15}$$

5 $\sin(2A)=\sin(A+A)$

$$=\sin A\cos A+\cos A\sin A$$

$$=\left(\frac{\sqrt{21}}{5}\right)\left(\frac{2}{5}\right)+\left(\frac{2}{5}\right)\left(\frac{\sqrt{21}}{5}\right)=\frac{4\sqrt{21}}{25}$$

6 $\cos(2B)=\cos(B+B)$

$$=\cos B\cos B-\sin B\sin B$$

$$=\left(\frac{2\sqrt{2}}{3}\right)\left(\frac{2\sqrt{2}}{3}\right)-\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)=\frac{7}{9}$$