4 Reduction to linear form

Try this worksheet after you have completed section 4.7

In Chapter 4 you looked at techniques for solving problems of the form $a^x = b$.
In this extension you will learn about a graphical application.

If an exponential relationship exists between two sets of data then you can use logarithms to simplify their exponential graph into a straight line.
This technique is called **reduction to linear form**.

EXAMPLE 1

In the equation $y = ab^x$, a and b are constants.
Reduce the equation $y = ab^x$ to linear form.

Answer

$\log y = \log ab^x$	*Take logs of both sides*
$\log y = \log a + \log b^x$	*$\log(ab) = \log a + \log b$*
$\log y = \log a + x \log b$	*$\log b^x = x \log b$*
$\log y = x \log b + \log a$	*Rearrange*

The equation is now in the form $Y = mX + c$, where $Y = \log y$, $X = x$, $m = \log b$ and $c = \log a$

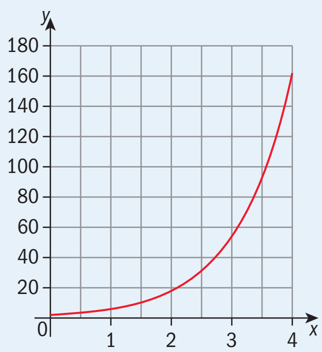
So if you plot $\log y$ vertically against x horizontally you will get a straight line with gradient $\log b$ and vertical intercept $\log a$.

EXAMPLE 2

The relationship between the values of x and y in the table is thought to be $y = ab^x$, where a and b are positive integers. Find a and b.

x	0	1	2	3	4
y	2	6	18	54	162

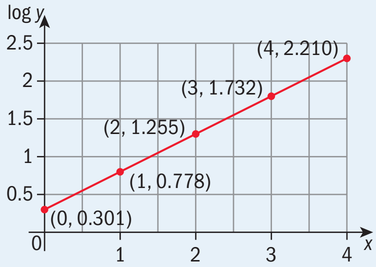
Answer



Plot the data on a graph. You get an exponential style curve.

Continued on next page

x	0	1	2	3	4
$\log y$	0.301	0.778	1.255	1.732	2.210



Plot log y against x – this gives a straight line.

From the table the intercept with the y-axis is 0.301

The value of log y when $x = 0$.

The gradient is approximately

$\frac{2.210 - 0.301}{4 - 0} = \frac{1.909}{4} = 0.47725$

Work out the gradient from two points on the graph.

$\log b \approx 0.47725$ and $b \approx 10^{0.47725}$ and so $b = 3$

$\log a \approx 0.301$ and $a \approx 10^{0.301}$ and so $a = 2$

So the equation that fits the data is $y = 2 \times 3^x$

$y = ab^x$ can be written in the linear form $\log y = x \log b + \log a$

$\log b$ is the gradient of the line.

$\log a$ is the y-intercept.

You can use a similar technique to solve equations of the form $y = ax^b$.
You can write the equation in the form $\log y = b\log x + \log a$ and in this case when $\log x$ is plotted horizontally and $\log y$ is plotted vertically the resulting straight line has a gradient of b and a vertical intercept of $\log a$.

Exercise

1 The relationship between x and y in this table is $y = ax^b$ where a and b are positive integers. Find a and b.

x	1	2	3	4
y	2	16	54	128

2 The population, P thousands, of a town is tabulated for 5 consecutive years as shown in this table.

t	1	2	3	4	5
P	12.8	20.5	32.8	52.4	83.9

a It is believed that P and t are connected by an exponential relationship of the form $P = ab^t$, where a and b are constants.
Verify this graphically and find the values of a and b.

b When is the population expected to exceed half a million?

3 An experiment is conducted to determine the relationship between the resistance to motion, R newtons, of a wooden board towed through water, and its speed, V ms^{-1}. The data were recorded and are shown in the table.

V	1.8	3.7	5.5	9.2	11
R	2.01	7.32	15.1	38.9	56.2

Assuming that $R = kV^n$, where n and k are constants

a obtain a relation between $\log R$ and $\log V$

b by drawing a graph of $\log R$ against $\log V$ estimate, to 2 significant figures, the values of n and k.

4 The period of oscillation, T seconds, of a heavy weight attached to a wire of length l metres was investigated for $2 \le l \le 8$. The results are shown in the table.

l	2	3	4	5	6	7	8
T	2.81	3.47	4.01	4.49	4.98	5.29	5.63

It is believed that l and T are related by an equation of the form $T = kl^n$ where k and n are positive constants.

a Plot values of $\log T$ against $\log l$ and hence state, giving reasons, whether or not l and T are related by an equation of this form.

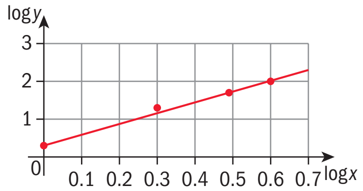
b Estimate the values of k and n.

Chapter 4 extension worked solutions

1 $y = ax^b \Rightarrow \log y = \log a + b\log x$

Plotting log x against log y gives a straight line verifying the rule.

log *x*	0	0.30	0.48	0.60
log *y*	0.3	1.20	1.73	2.11



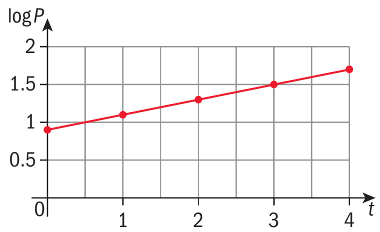
y-intercept = 0.3 to 1 sf so $\log a \approx 0.3 \Rightarrow a \approx 10^{0.3} \approx 2$

gradient $= \dfrac{1.81}{0.6} = 3.0$ (2 sf) so $b = 3$

$\therefore y = 2x^3$

2 **a** $P = ab^t \Rightarrow \log P = \log a + t\log b$

Plotting t against log P gives a straight line verifying the rule.



y-intercept = 0.9 to 1 sf, so $\log a \approx 0.9 \Rightarrow a \approx 10^{0.9} = 7.9$ (2 sf)

gradient = 0.2 so $\log b \approx 0.2 \Rightarrow b \approx 10^{0.2} = 1.6$ (2 sf)

$\therefore P = 7.9 \times 1.6^t$

b Population exceeds half a million when

$$7.9 \times 1.6^t = 500$$

$$1.6^t = \left(\frac{500}{7.9}\right)$$

$$t \log 1.6 = \log\left(\frac{500}{7.9}\right)$$

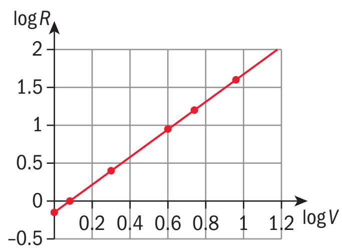
$$t = \frac{\log\left(\frac{500}{7.9}\right)}{\log 1.6}$$

$$t = 8.8 \text{ (2 sf)}$$

Therefore after 8.8 years.

3 **a** $R = kV^n \Rightarrow \log R = \log k + n\log V$

b Plotting log V against log R gives a straight line verifying the rule.



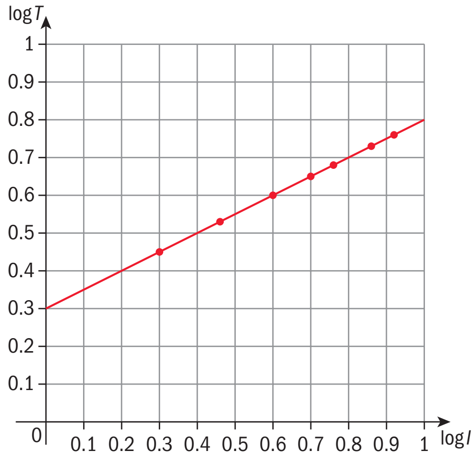
y-intercept = −0.17, so so $\log k \approx -0.17 \Rightarrow k \approx 10^{-0.17} \approx 0.68$ (2 sf)

gradient = 1.84, so $n = 1.84$

$R = 0.68V^{1.84}$

4 **a** $T = kl^n \Rightarrow \log T = \log k + n\log l$

Plotting $\log l$ against $\log T$ gives a straight line verifying the rule.



b y-intercept = 0.3 so $\log k \approx 0.3 \Rightarrow k \approx 10^{0.3} \approx 2$

gradient = 0.5 so $n \approx 0.5$

$\therefore T = 2\sqrt{l}$