1 Polynomials

Try the first section of this worksheet after you have completed section 1.3, and the second section after you have completed Exercise 1H.

Dividing polynomials

To divide a polynomial by a divisor more complex than a simple monomial you use a method called polynomial long division. This is just like long division with numbers, except that you divide with variables.

EXAMPLE 1

If $f(x) = 2x^2 + 3x + 4$ and $g(x) = x + 2$, find $\frac{f(x)}{g(x)}$

Answer

$\frac{f(x)}{g(x)} = \frac{2x^2+3x+4}{x+2}$

Divide $2x^2 + 3x + 4$ by $x + 2$

$$\begin{array}{r} 2x-1 \\ x+2\,\overline{)\,2x^2+3x+4} \\ \underline{2x^2+4x} \\ -x+4 \\ \underline{-x-2} \\ 6 \end{array}$$

1 *Start by dividing $2x^2$ by x to give $2x$*

2 *Multiply $2x$ by the divisor $(x + 2)$ to give $2x^2 + 4x$*
Write this below the terms of the same power in the dividend.

3 *Underline and subtract to get $-x$ and bring down the next term $(+4)$*

Repeat step 1, this time dividing $-x$ by x to give -1. Write -1 in the answer line.

Repeat step 2: $-1 \times (x + 2) = -x - 2$

Repeat step 3: subtract $-x - 2$ from $-x + 4 = 6$

The remainder is 6

$\frac{f(x)}{g(x)} = \frac{2x^2+3x+4}{x+2}$

$= 2x - 1$ remainder 6

or $\frac{f(x)}{g(x)} = 2x - 1 + \frac{6}{x+2}$

You can write the remainder over the divisor to give a more complete answer.

The top and bottom of a fraction in a division are called the $\frac{\text{dividend}}{\text{divisor}}$

You can check your answers by multiplying the divisor by the quotient and adding the remainder to get the dividend

Exercise 1

Find $\frac{f(x)}{g(x)}$ for each pair of functions.

1 $f(x) = 2x^2 + 10x + 4$ and $g(x) = x - 3$

2 $f(x) = 3x^2 - 5x + 1$ and $g(x) = x + 2$

3 $f(x) = 2x^2 - 3x + 8$ and $g(x) = 2x + 1$

4 $f(x) = 4x^3 + 4x^2 - x + 5$ and $g(x) = 2x - 3$

5 $f(x) = 6x^3 + 4x^2 + 5$ and $g(x) = 3x - 1$

6 $f(x) = 2x^4 + 3x^3 + 5x - 1$ and $g(x) = x^2 - 2x + 2$

5 The coefficient of x is zero so you must write the dividend as $6x^3 + 4x^2 + 0x + 5$

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Investigation polynomial division

Try this after you have completed Exercise 1H.
Let $f(x) = 2x^3 + 11x^2 + 18x + 9$

1 Use long division to find the remainder when $f(x)$ is divided by $x - 2$
2 Now find $f(2)$.
3 Use long division to find the remainder when $f(x)$ is divided by $x + 4$
4 Now find $f(-4)$.
5 **a** How are $f(2)$ and $f(-4)$ related to the remainders?
b Copy and complete:
When a polynomial $f(x)$ is divided by $(x - a)$ the remainder is _____.

This statement is called the **Remainder theorem** in further algebra courses.

6 Use long division to find the remainder when $f(x)$ is divided by
a $x + 1$ **b** $x + 3$ **c** $2x + 3$
7 Find **a** $f(-1)$ **b** $f(-3)$ **c** $f\left(-\frac{3}{2}\right)$

→ If the remainder is zero the divisor a **factor** of the dividend.

8 Copy and complete:
When a polynomial $f(x)$ is divided by $(x - a)$ the remainder is ___ if and only if $(x - a)$ is a factor of $f(x)$.

This is also part of the **Remainder theorem**.

9 Now multiply $(x + 1)(x + 3)(2x + 3)$ together.
10 What do you notice?
11 Challenge 1
Given that $(x - 2)$ is a factor of $x^3 - 2x^2 - 9x + 18$, find the other two factors of the expression.
12 Challenge 2
Find the factors of $x^3 - 7x - 6$

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Chapter 1 extension worked solutions

Exercise 1

1

$$\begin{array}{r|r} & 2x+16 \\ \cline{2-2} x-3 & 2x^2+10x+4 \\ & 2x^2-6x \\ \cline{2-2} & 16x+4 \\ & 16x-48 \\ \cline{2-2} & 52 \end{array}$$

$$\frac{2x^2+10x+4}{x-3}=2x+16+\frac{52}{x-3}$$

2

$$\begin{array}{r|r} & 3x-11 \\ \cline{2-2} x+2 & 3x^2-5x+1 \\ & 3x^2+6x \\ \cline{2-2} & -11x+1 \\ & -11x-22 \\ \cline{2-2} & 23 \end{array}$$

$$\frac{3x^2-5x+1}{x+2}=3x-11+\frac{23}{x+2}$$

3

$$\begin{array}{r|r} & x-2 \\ \cline{2-2} 2x+1 & 2x^2-3x+8 \\ & 2x^2+x \\ \cline{2-2} & -4x+8 \\ & -4x-2 \\ \cline{2-2} & 10 \end{array}$$

$$\frac{2x^2-3x+8}{2x+1}=x-2+\frac{10}{2x+1}$$

4

$$\begin{array}{r|r} & 2x^2+5x+7 \\ \cline{2-2} 2x-3 & 4x^3+4x^2-x+5 \\ & 4x^3-6x^2 \\ \cline{2-2} & 10x^2-x+5 \\ & 10x^2-15x \\ \cline{2-2} & 14x+5 \\ & 14x-21 \\ \cline{2-2} & 26 \end{array}$$

$$\frac{4x^3+4x^2-x+5}{2x-3}=2x^2+5x+7+\frac{26}{2x-3}$$

5

$$\begin{array}{r|r} & 2x^2+2x+\frac{2}{3} \\ \cline{2-2} 3x-1 & 6x^3+4x^2+0x+5 \\ & 6x^3-2x^2 \\ \cline{2-2} & 6x^2+5 \\ & 6x^2-2x \\ \cline{2-2} & 2x+5 \\ & 2x-\frac{2}{3} \\ \cline{2-2} & \frac{17}{3} \end{array}$$

$$\frac{6x^3+4x^2+5}{3x-1}=2x^2+2x+\frac{2}{3}+\frac{\frac{17}{3}}{3(3x-1)}$$

6

$$\begin{array}{r|r} & 2x^2+7x+10 \\ \cline{2-2} x^2-2x+2 & 2x^4+3x^3+0x^2+5x-1 \\ & 2x^4-4x^3+4x^2 \\ \cline{2-2} & 7x^3-4x^2+5x-1 \\ & 7x^3-14x^2+14x \\ \cline{2-2} & 10x^2-9x-1 \\ & 10x^2-20x+20 \\ \cline{2-2} & 11x-21 \end{array}$$

$$\frac{2x^4+3x^3+5x-1}{x^2-2x+2}=2x^2+7x+10+\frac{11x-21}{x^2-2x+2}$$

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Investigation

1

$$
\begin{array}{r|l}
 & 2x^2 + 15x + 48 \\ \cline{2-2}
x-2 & 2x^3 + 11x^2 + 18x + 9 \\
 & 2x^3 - 4x^2 \\ \cline{2-2}
 & 15x^2 + 18x + 9 \\
 & 15x^2 - 30x \\ \cline{2-2}
 & 48x + 9 \\
 & 48x - 96 \\ \cline{2-2}
 & 105
\end{array}
$$

$$\frac{2x^3+11x^2+18x+9}{x-2} = 2x^2+15x+48+\frac{105}{x-2}$$

The remainder is 105.

2 $f(2) = 2(2)^3 + 11(2)^2 + 18(2) + 9$

$f(2) = 16 + 44 + 36 + 9 = 105$

3

$$
\begin{array}{r|l}
 & 2x^2 + 3x + 6 \\ \cline{2-2}
x+4 & 2x^3 + 11x^2 + 18x + 9 \\
 & 2x^3 + 8x^2 \\ \cline{2-2}
 & 3x^2 + 18x + 9 \\
 & 3x^2 + 12x \\ \cline{2-2}
 & 6x + 9 \\
 & 6x + 24 \\ \cline{2-2}
 & -15
\end{array}
$$

The remainder is −15.

4 $f(-4) = 2(-4)^3 + 11(-4)^2 + 18(-4) + 9$

$f(-4) = -128 + 176 - 72 + 9 = -15$

5 **a** $f(2)$ and $f(-4)$ have the same values as the remainders.

b When a polynomial $f(x)$ is divided by $(x - a)$, the remainder is $f(a)$.

6 **a**

$$
\begin{array}{r|l}
 & 2x^2 + 9x + 9 \\ \cline{2-2}
x+1 & 2x^3 + 11x^2 + 18x + 9 \\
 & 2x^3 + 2x^2 \\ \cline{2-2}
 & 9x^2 + 18x + 9 \\
 & 9x^2 + 9x \\ \cline{2-2}
 & 9x + 9 \\
 & 9x + 9 \\ \cline{2-2}
\end{array}
$$

The remainder is zero.

b

$$
\begin{array}{r|l}
 & 2x^2 + 5x + 3 \\ \cline{2-2}
x+3 & 2x^3 + 11x^2 + 18x + 9 \\
 & 2x^3 + 6x^2 \\ \cline{2-2}
 & 5x^2 + 18x + 9 \\
 & 5x^2 + 15x \\ \cline{2-2}
 & 3x + 9 \\
 & 3x + 9 \\ \cline{2-2}
\end{array}
$$

The remainder is zero.

c

$$
\begin{array}{r|l}
 & x^2 + 4x + 3 \\ \cline{2-2}
2x+3 & 2x^3 + 11x^2 + 18x + 9 \\
 & 2x^3 + 3x^2 \\ \cline{2-2}
 & 8x^2 + 18x + 9 \\
 & 8x^2 + 12x \\ \cline{2-2}
 & 6x + 9 \\
 & 6x + 9 \\ \cline{2-2}
\end{array}
$$

The remainder is zero.

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7 **a** $f(-1) = 2(-1)^3 + 11(-1)^2 + 18(-1) + 9$

$f(-1) = 2 + 11 - 18 + 9 = 0$

b $f(-3) = 2(-3)^3 + 11(-3)^2 + 18(-3) + 9$

$f(-3) = -54 + 99 - 54 + 9 = 0$

c $f\left(-\frac{3}{2}\right) = 2\left(-\frac{3}{2}\right)^3 + 11\left(-\frac{3}{2}\right)^2 + 18\left(-\frac{3}{2}\right) + 9$

$f\left(-\frac{3}{2}\right) = -\frac{27}{4} + \frac{99}{4} - 27 + 9 = 0$

8 When a polynomial $f(x)$ is divided by $(x - a)$, the remainder is **zero** if and only if $(x - a)$ is a factor of $f(x)$.

9 $(x + 1)(x + 3)(2x + 3) = f(x) = 2x^3 + 11x^2 + 18x + 9$

10 Multiplying the factors results in the original function.

11
$$
\begin{array}{r|l}
 & x^2 \qquad -9 \\ \hline
x-2 & x^3 - 2x^2 - 9x + 18 \\
 & \underline{x^3 - 2x^2} \\
 & \qquad -9x + 18 \\
 & \qquad \underline{-9x + 18}
\end{array}
$$

$x^2 - 9 = (x + 3)(x - 3)$

12 $x^3 - 7x - 6$

Find $f(-1)$:

$f(-1) = (-1)^3 - 7(-1) - 6 = 0$

Therefore $(x + 1)$ is a factor.

Divide $f(x)$ by $(x + 1)$

$$
\begin{array}{r|l}
 & x^2 - x - 6 \\ \hline
x+1 & x^3 + 0x^2 - 7x - 6 \\
 & \underline{x^3 + x^2} \\
 & \quad -x^2 - 7x - 6 \\
 & \quad \underline{-x^2 - x} \\
 & \qquad -6x - 6 \\
 & \qquad \underline{-6x - 6}
\end{array}
$$

Now factorize $x^2 - x - 6$ to give $(x - 3)(x + 2)$.

The factors of $f(x)$ are $(x + 1)(x - 3)(x + 2)$.

Notice that $-1 + 7 - 6 = 0$ or try different values of x until you find a suitable value. Often $1, -1, 2, -2, 3$ or -3 will be one of the values.

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