9 More volumes of solids of revolution

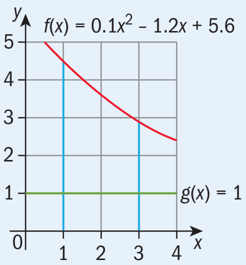
Try this worksheet after you have completed Section 9.6

In Chapter 9 you found the volume of a solid of revolution formed by rotating a plane figure about the x-axis. In each case the plane figure was bordered by the axis of rotation and you used the disc method to find the volume.
This worksheet looks at rotating a plane figure about the x-axis when the figure is not bordered by the x-axis.

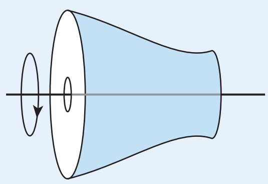
EXAMPLE 1

Find the volume of the solid formed by revolving the region bounded by the graphs of $f(x) = 0.1x^2 - 1.2x + 5.6$, $g(x) = 1$, $x = 1$ and $x = 3$ about the x-axis.

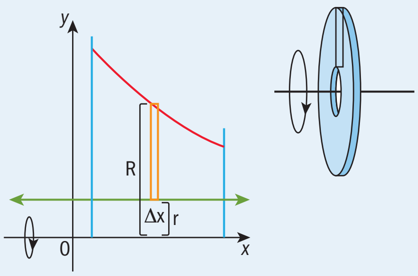
Answer



Sketch the graphs on your GDC. The solid will look something like this.



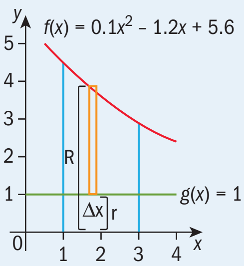
If you rotate a representational rectangle in the region about the x-axis, you won't get a disc, rather you create a '**washer**' that is a cylinder with a cylindrical hole in the center.



The volume of this 'washer' is given by $\pi R^2 \mathrm{d}x - \pi r^2 \mathrm{d}x$ or $\pi\left(R^2 - r^2\right)\mathrm{d}x$.

In this case $R = 0.1x^2 - 1.2x + 5.6$ and $r = 1$ and so:

$$V = \int_1^3 \pi\left[\left(0.1x^2 - 1.2x + 5.6\right)^2 - 1^2\right]\mathrm{d}x$$
$$= 78.0 \text{ (3 sf)}$$

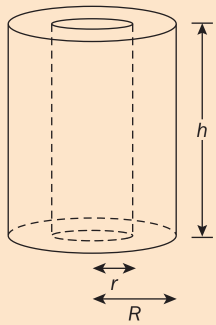


The 'washer' formula is based on the volume of a cylinder:

$V = \pi r^2 h$

$V = \pi R^2 h - \pi r^2 h$

$= \pi\left(R^2 - r^2\right)h$

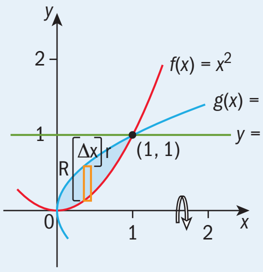


In Example 2 a plane figure rotates about a horizontal line that is not the x-axis.

EXAMPLE 2

Find the volume of the solid formed by revolving the region bounded by the graphs of $f(x) = x^2$ and $g(x) = \sqrt{x}$ about the line $y = 1$.

Answer



$$V = \int_0^1 \pi\left[\left(1 - x^2\right)^2 - \left(1 - \sqrt{x}\right)^2\right]dx$$
$$= 1.15 \text{ (3 sf)}$$

R is the distance between the horizontal line $y = 1$ and f, so $R = 1 - x^2$

r is the distance between the horizontal line $y = 1$ and g, so $r = 1 - \sqrt{x}$

The functions intersect at $x = 0$ and $x = 1$, so the limits of integration are from 0 to 1.

Exercise

Find the volume of the solid formed when the region bounded by the graphs of the functions is rotated around the given axis.

1 $f(x) = 6 - x^2$, $g(x) = 2$

a the x-axis **b** the line $y = 1$ **c** the line $y = 8$

2 $f(x) = 4x - x^2$, $g(x) = x^2$

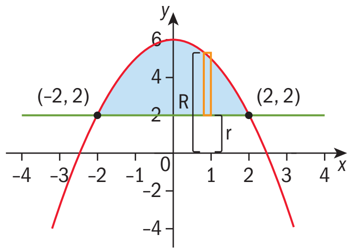
a the x-axis **b** the line $y = 4$ **c** the line $y = -1$

Chapter 9 extension worked solutions

1 $f(x) = 6 - x^2$, $g(x) = 2$

a the x-axis

$$V = \int_{-2}^{2} \pi\left[(6 - x^2)^2 - 2^2\right]\mathrm{d}x = 241 \text{ (3 sf)}$$

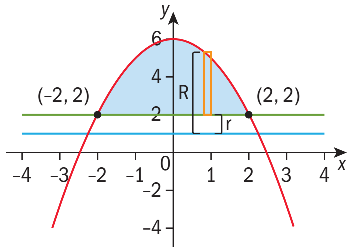


b the line $y = 1$

$R = (6 - x^2) - 1 = 5 - x^2$

$r = 2 - 1 = 1$

$$V = \int_{-2}^{2} \pi\left[(5 - x^2)^2 - 1^2\right]\mathrm{d}x = 174 \text{ (3 sf)}$$

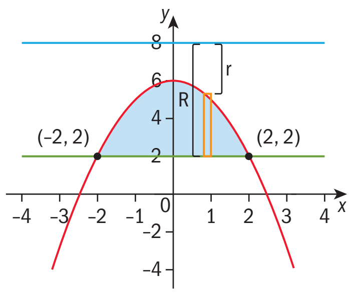


c the line $y = 8$

$R = 8 - 2 = 6$

$r = 8 - (6 - x^2) = 2 + x^2$

$$V = \int_{-2}^{2} \pi\left[6^2 - (2 + x^2)^2\right]\mathrm{d}x = 295 \text{ (3 sf)}$$

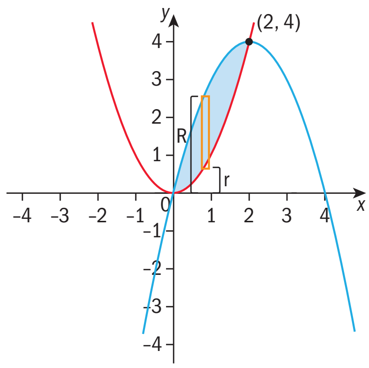


2 $f(x) = 4x - x^2$, $g(x) = x^2$

a the x-axis

$$V = \int_{0}^{2} \pi\left[(4x - x^2)^2 - (x^2)^2\right]\mathrm{d}x$$

$= 33.5$ (3 sf)



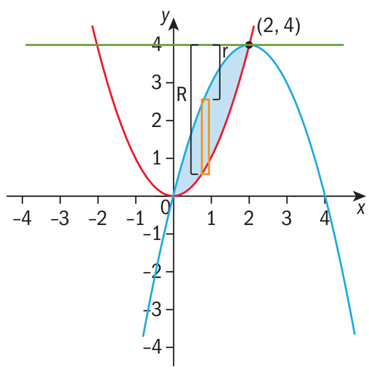
b the line $y = 4$

$R = 4 - x^2$

$r = 4 - (4x - x^2) = 4 - 4x + x^2$

$$V = \int_{0}^{2} \pi\left[(4 - x^2)^2 - \left(4 - 4x + x^2\right)^2\right]\mathrm{d}x$$

$= 33.6$ (3 sf)



c the line $y = -1$

$R = (4x - x^2) - (-1) = 1 + 4x - x^2$

$r = x^2 - (-1) = x^2 + 1$

$$V = \int_{0}^{2} \pi\left[(1 + 4x - x^2)^2 - \left(x^2 + 1\right)^2\right]\mathrm{d}x$$

$= 52.3$ (3 sf)